Comments on an Experiment on Transverse Coalescing

D. A. Edwards, Deutsches Elektronen Synchrotron DESY,
M. J. Syphers, Fermi National Accelerator Laboratory*

August 8, 2008

Abstract

A method of coalescing a particle distribution that occupies a ring in the phase space of one transverse degree-of-freedom has been proposed by Derbenev. [1] The purpose of this note is to outline a potential experimental realization of the technique using the electron beam of a photoinjector.

1 Introduction

The process of adiabatic capture into RF buckets has a very interesting analog in the transverse plane - another invention of Derbenev's [1]. Suppose one starts with a ring beam in transverse configuration space, such as may be produced by a ring laser spot on a photocathode. Then put the bunch through a round-to-flat transformer[2] so that one has a ring in the phase space of one of the transverse degrees of freedom. So now in that degree-of-freedom, there is a distribution with some width Δa in amplitude and uniformly populated in angle ϕ.

Now suppose the bunch enters a focusing channel in which there is an octopole term so that the phase advance is amplitude dependent and becomes an integer multiple of 2π at an amplitude a_0. Steering dipoles are installed to produce deflections in synchronism with the phase advance at the amplitude a_0, and the dipole strength increases gradually along the channel, mimicking the adiabatic turn-on of the RF in the longitudinal case. In synchrotron language, there is a driven integer resonance at amplitude a_0.

The point of this note is to see if this process can be demonstrated in one of today's photoinjectors at reasonable cost, as another of the phase-space manipulations that may play a role in future injection systems. Incorporation of the slow increase of the dipole terms suggests that a ring rather than a linear channel is the most reasonable structure, but the discussion will be carried on for a while without making that choice.

*The Fermi National Accelerator Laboratory is operated under contract with the US Department of Energy

2 Parameter Definition

Let L be the distance along the central trajectory of the structure over which the phase of a (linear) oscillation would advance by almost 2π, increasing with amplitude to 2π at amplitude a_0 due to the octopole driven tune-shift. Let $\Delta B(x,s)$ represent the dipole and octopole fields, where x is the transverse displacement from and s distance along the central trajectory. Let ψ represent the phase modulo 2π at some periodic points of observation separated by L in the structure.

Then, proceeding in the spirit of a perturbation of the linear motion, the change in amplitude resulting from a single passage of the distance L is[3]

$$\Delta a = \frac{\beta_0}{(B\rho)} \int_0^L \left(\frac{\beta(s)}{\beta_0}\right)^{1/2} \Delta B(x,s) \sin[\psi + \phi(s)] ds \tag{1}$$

where β_0 is the amplitude function at the point of observation, $(B\rho)$ is the magnetic rigidity, and the unperturbed motion is described by

$$x(s) = a \left(\frac{\beta(s)}{\beta_0}\right)^{1/2} \cos[\psi + \phi(s)]. \tag{2}$$

Using subscripts to represent the number of poles in a field contribution, write $\Delta B = \Delta B_2 + \Delta B_8$. Then for the integer resonance driving field take

$$\Delta B_2 = -\left(\frac{\beta_0}{\beta(s)}\right)^{3/2} B_0 \cos\phi(s) \tag{3}$$

where B_0 is a positive constant. The reason for the explicit choice of sign will be apparent later. Setting $ds = \beta(s) d\phi$ in Eq. 1, one finds

$$\Delta a_2 = -\frac{\pi\beta_0^2}{(B\rho)} B_0 \sin\psi = -\frac{\pi\beta_0}{L}\beta_0\Theta_2 \sin\psi. \tag{4}$$

In the rightmost expression of Eq. 4, the angle Θ_2 is the deflection that would be produced if B_0 extended a distance L; $\Theta_2 \equiv B_0 L/(B\rho)$. This expression looks reasonable; the factor $\pi\beta_0/L$ is typically of unit order of magnitude while $\Theta_2\beta_0$ is the characteristic change in amplitude due to the angular deflection Θ_2.

The octopole field must be chosen so that it does not contribute to Δa. This can be accomplished by the choice

$$\Delta B_8 = \left(\frac{B'''(s)}{6}\right) x^3 = \left(\frac{\beta_0}{\beta}\right)^3 \left(\frac{B'''}{6}\right)_0 x^3. \tag{5}$$

Then

$$\Delta a_8 = \frac{\beta_0^2}{(B\rho)} \left(\frac{B'''}{6}\right)_0 a^3 \int \sin[\psi + \phi] \cos^3[\psi + \phi] d\phi = 0 \tag{6}$$

as desired.

The treatment for phase advance follows in an analogous fashion, starting from

$$\Delta\psi = \frac{\beta_0}{(B\rho)} \int_0^L \left(\frac{\beta(s)}{\beta_0}\right)^{1/2} \frac{\Delta B(x,s)}{a} \sin[\psi + \phi(s)] ds. \tag{7}$$

For the dipole contribution,

$$\Delta\psi_2 = -\frac{\pi\beta_0^2}{(B\rho)} \frac{B_0}{a} \cos\psi = -\frac{\pi\beta_0}{L} \frac{\beta_0\Theta_2}{a} \cos\psi. \tag{8}$$

In the case of the octopole, the integral no longer vanishes but yields $3\pi/4$ with the result

$$\Delta\psi_8 = \frac{3}{4} \frac{\pi\beta_0^2}{(B\rho)} \left(\frac{B'''}{6}\right)_0 a^2. \tag{9}$$

Now modify Eq. 9 in two ways. Replace a^2 by $a^2 - a_0^2$ in order to reference the rotation in phase space to the radius a_0. And in a similar fashion to the use of Θ_2, define the corresponding quantity for the octopole field as the total angular deflection at radius a_0. The result is

$$\Delta\psi_8 = \frac{3}{4} \frac{\pi\beta_0}{L} \frac{\beta_0\Theta_8}{a_0^3} (a^2 - a_0^2), \tag{10}$$

where $\Theta_8 \equiv (B'''/6)_0\, a_0^3 L/(B\rho)$.

To reflect the ramp of the dipole field, place a factor b with $0 \le b \le 1$ in front of Θ_2. Set $a = ra_0$. Finally, interpreting Δa and $\Delta\psi$ as derivatives $da/dn = a_0 dr/dn$ and $d\psi/dn$ gives the set of differential equations

$$\frac{dr}{dn} = -k\Theta_2 b \sin\psi \tag{11}$$

$$\frac{d\psi}{dn} = k\left[\frac{3}{4}\Theta_8(r^2-1) - \frac{1}{r}\Theta_2 b\cos\psi\right] \tag{12}$$

$$k \equiv \frac{\pi\beta_0}{L}\frac{\beta_0}{a_0}. \tag{13}$$

Eqs. 11, 12 are of the form of Hamilton's equations,

$$\frac{dr}{dn} = -\frac{1}{r}\frac{\partial H}{\partial\psi}, \quad \frac{d\psi}{dn} = \frac{1}{r}\frac{\partial H}{\partial r} \tag{14}$$

for the Hamiltonian

$$H = k\left[\frac{3\Theta_8}{16}(r^2-1)^2 - \Theta_2 br\cos\psi\right]. \tag{15}$$

Fixed points are at $\psi = 0$ and $\psi = \pi$, stable and unstable respectively. Placement of the stable fixed point at the "below transition" location was the motivation for the sign choice in Eq. 3. From the expression for the separatrix passing through $r = 1$, $\psi = \pi$, the width of the resonance island at $\psi = 0$ is

$$w = 2\left(\frac{8}{3}\frac{\Theta_2}{\Theta_8}\right)^{1/2}. \tag{16}$$

Expansion of Eqs. 11 and 12 in the neighborhood of the stable fixed point gives, after combination into a second order equation for ψ:

$$\frac{d^2\psi}{dn^2} + (2\pi\nu_s)^2\psi = 0, \quad \nu_s \equiv \left[\frac{3}{32\pi^2}k^2\Theta_2\Theta_8 b\right]^{1/2}. \tag{17}$$

The quantity ν_s, the analog of the synchrotron oscillation tune, presumably must be sufficiently small compared with unity to satisfy the adiabaticity requirement.

As input parameters, let us choose L, a_0, w, and $\Delta\phi$, this last being the difference in phase advance between amplitude a_0 and small oscillations. With use of Eq. 12,

$$\Theta_8 = \frac{4La_0}{3\pi\beta_0^2}\Delta\phi \tag{18}$$

and from Eq. 16

$$\Theta_2 = \frac{3}{32}\Theta_8 w^2. \tag{19}$$

The relationship between β_0 and L depends on the focusing structure. For a weak focusing ring of circumference L and field index n_f, $\beta_0 = L/[2\pi\sqrt{(1 - n_f)}]$, while for a sequence of FODO cells, $\beta_0 \approx (2 + \sqrt{2})L/8$.

3 Channel Example

As an example, suppose the channel is a ring bending magnet of the sort that might be used in a weak focusing synchrotron. The small amplitude tune, ν, is somewhat less than one, and an octopole term constant in azimuth produces the integer tune at the appropriate amplitude. The dipole kick is provided at a single location on the ring. Of course, this simple example makes no provision for injection or extraction.

The equation of motion for transverse oscillations of a particle about the beam axis within the bend magnet is

$$\frac{d^2u}{dn^2} = -(2\pi\nu)^2 u - \frac{L\Theta_8}{a_0}u^3. \tag{20}$$

A computer code was written using `R`.[4] To illustrate the process, we envisage the synchrotron described above and model it as m sections of length L/m with octupole "kicks" in the middle of each section. At one of the section interfaces in the ring a ramped dipole kick is provided. Defining $u \equiv x/a_0$, $v \equiv \beta_0 x'/a_0$, and $\vec{u} \equiv (u,v)^T$, we cast Eq. 20 into matrix form,

$$\vec{u}_{i+1} = R[R\vec{u}_i + \vec{\Theta}(R\vec{u}_i)], \tag{21}$$

transporting $\vec{u}$ through the i^{th} section. The matrix R is a rotation matrix through angle $2\pi\nu/m/2$, and the function $\vec{\Theta}$ operating on the vector $(\vec{X}) = (x_1, x_2)^T$ is determined by the required octupole strength. In terms of Θ_8 the kick per section will be $\Delta v = \beta_0 \Delta x'/a_0 = (\beta_0/a_0)(\Theta_8/m)u^3 = \frac{8}{3}(\Delta\phi/m)u^3$, or

$$\Theta(\vec{X}) = \left(\begin{array}{c} 0 \\ -\frac{8\Delta\phi}{3m} x_1^3 \end{array} \right). \tag{22}$$

In the simulation, m is chosen to be 32.

Similarly, the dipole kick given once per revolution (once per m sections) is to generate $\Delta v = (\beta_0/a_0)\Theta_2$, or

$$\Delta\vec{u} = \left(\begin{array}{c} 0 \\ \frac{1}{4}\Delta\phi\ w^2 \end{array} \right). \tag{23}$$

In the code the dipole field is increased over N_b revolutions according to

$$b(n) = 1 - e^{-5(n/N_b)^2}. \tag{24}$$

To provide a smooth adiabatic ramp, N_b is chosen to be several thousand turns.

Using the parameters $L = 2$ m, $a_0 = 4 \times 10^{-3}$ m, $\Delta\phi = \pi/8$ and $w = 0.5$ the code produces the phase space plots shown in Figure 1. The initial distribution is a uniformly populated ring of 100 particles at $r = 1$, and the next three plots are for revolution numbers 1000, 2000, and 4000. Figure 2 shows the initial and final ($N_b = 10^4$) phase space plots, the dipole ramp function $b(n)$, and the variation of the rms values of the variables u and v during the process. As can be seen, the distribution settles down at about turn number 4-5000, where b has reached about 60-70% of its final value.

4 Discussion

As expected based on what we have been told by colleagues who have modeled the process[5], the initial ring distribution is reconfigured into a bunch-like shape. The behavior is indeed quite similar to the adiabatic capture process in the longitudinal degree-of-freedom.

Several measures are needed to turn this process into a candidate for construction. To accommodate injection, extraction, and diagnostics, one approach would be to turn the system into a racetrack by introduction of two straight sections equipped with quadrupoles for β-matching. In order to avoid introduction of octopole resonance driving terms, the phase advance of each such straight section should be 2π or a multiple thereof. Because of the use of the intrinsically sensitive integer tune, provision for correctors will likely be necessary.

No mention has been made of bunch length effects; in particular, the need for an isochronous structure. Suppose the momentum spread is at the $\Delta p/p \approx 10^{-3}$ level. Then for the weak focusing ring, a bunch of initially negligible length would spread into one wrapped several times around the circumference. The insertion of negative bends reminiscent of the stellarator approach may be used to address this problem. But inclusion of momentum spread into the model may uncover the need for correction of chromatic effects.

The large number of iterations used in the example of the preceding section is worrisome. The distribution settles down at some 5000 turns, but that is over 30 μs, which is a long time if a rather rapid cycling application is envisaged. A degree of miniaturization coupled with tighter resonance dipole spacing could be explored. But there is a qualitative distinction between this process and the other two phase space manipulations that we have had occasion to study, namely, the flat beam transformation cited earlier, and the longitudinal-transverse interchange method[6]. Both of those involve a single passage through a rather short structure.

Finally, despite the use of the term "coalesce", the technique conserves phase space, therefore any advantage in its use must be found in amelioration of space charge effects in the bunch generation steps.

References

[1] Ya. S. Derbenev, "Advanced optical concepts for electron cooling", Nucl. Instr. and Meth. A **441** (2000) 221-233. See the discussion beginning at the bottom of page 226.

[2] R. Brinkmann, Ya. Derbenev, K. Flöttmann, "A Flat Beam Electron Source for Linear Colliders", TESLA Note 99-09, April 1999.

[3] D. A. Edwards, M. J. Syphers, "Introduction to the Physics of High Energy Accelerators", J. Wiley and Sons, 1993. The material used here begins on page 116 of this textbook.

[4] The R Project for Statistical Computing, http://www.r-project.org/ .

[5] Private communications from R. Brinkmann, K. Flöttmann, S. Nagaitsev

[6] See, for example, the review article "Transverse-Longitudinal Phase-Space Manipulations and Correlations" by K. Kim and A. Sessler, from the proceedings of COOL05 published as AIP Conf. Proc. 821:115-138, 2006.

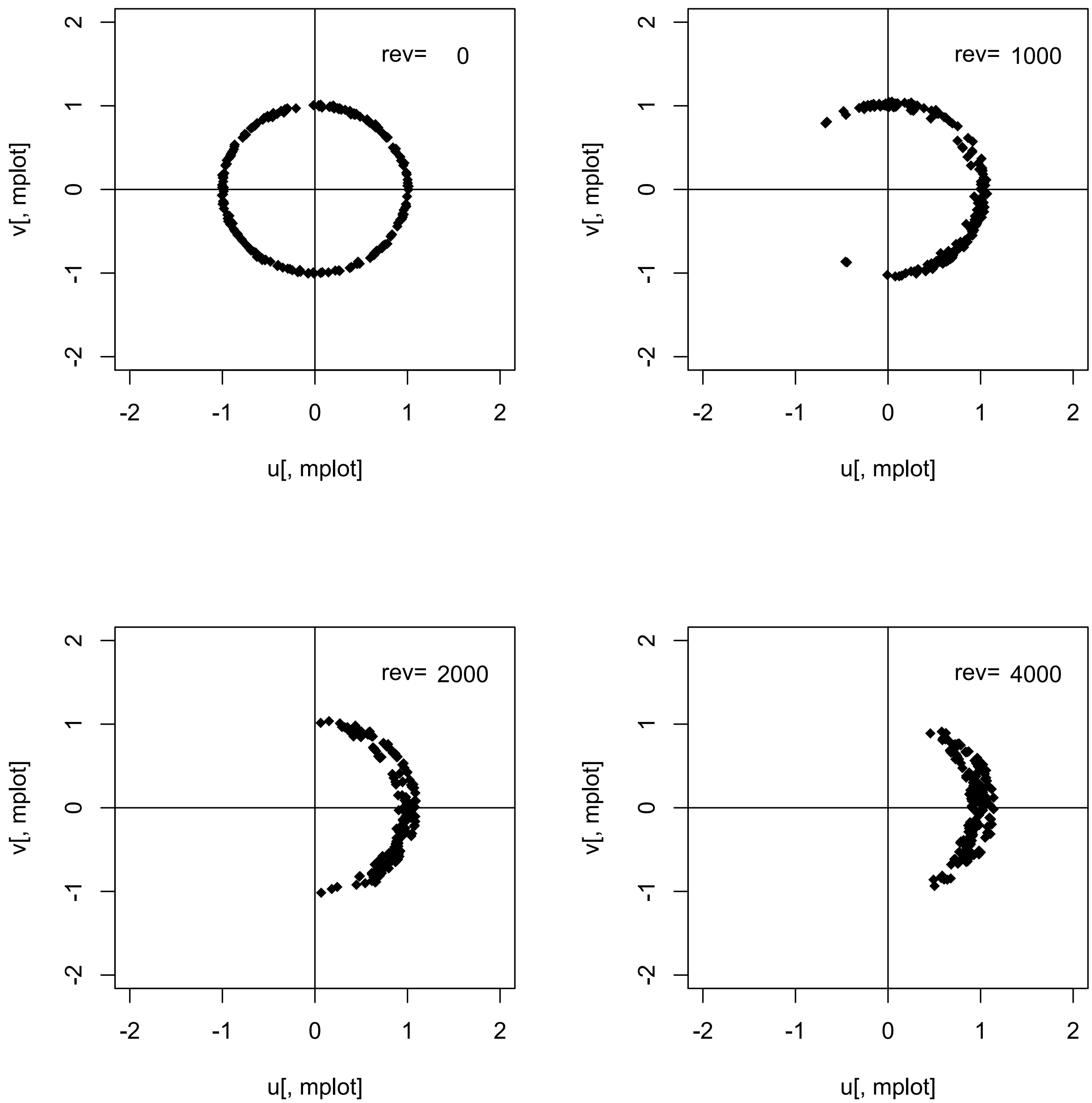

Figure 1: Phase space plots during adiabatic increase of dipole field. After 4000 revolutions, the deflection angle is 55% of Θ_2.

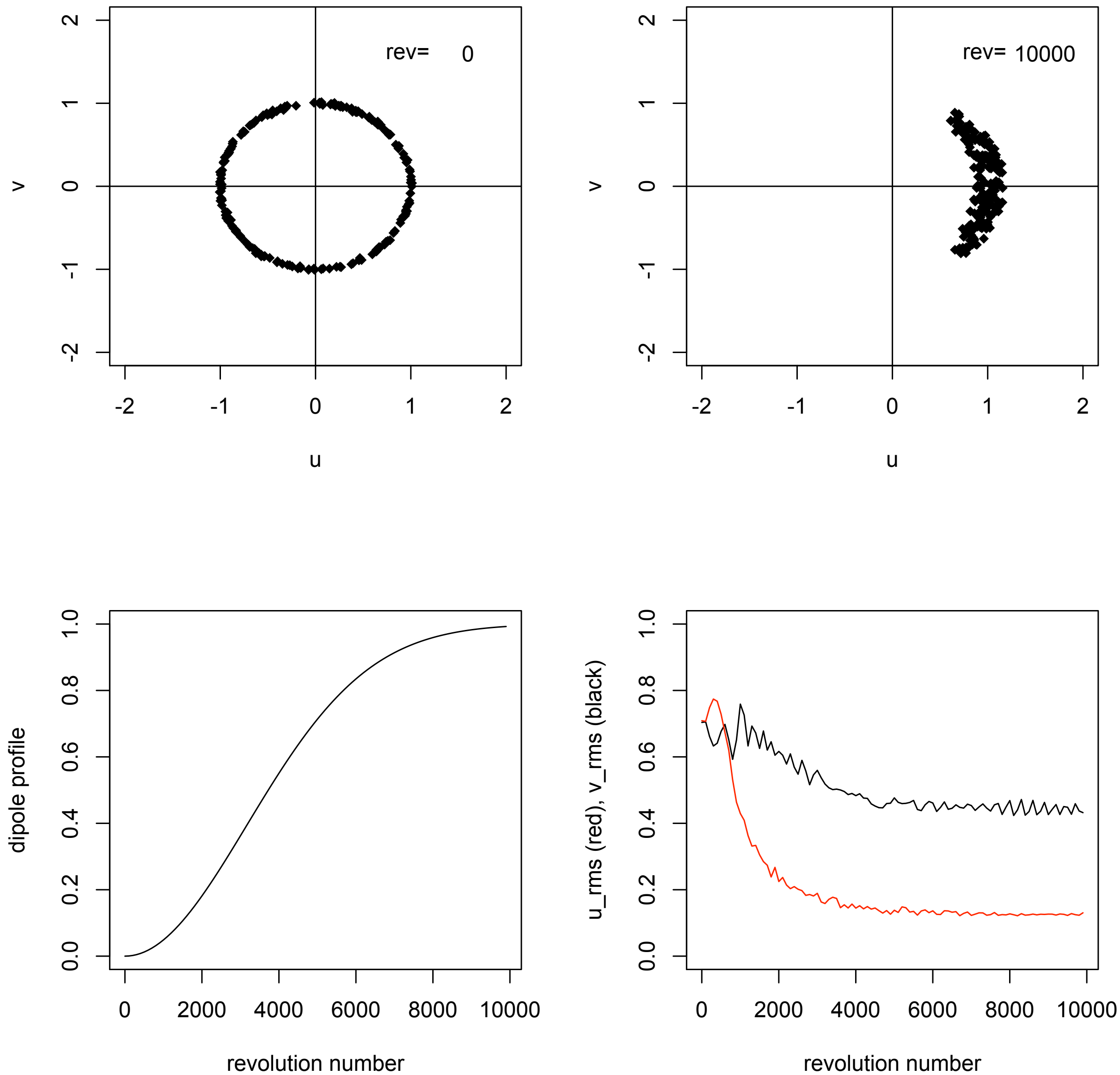

Figure 2: Top: Initial and final phase space plots; final value of dipole deflection angle is Θ_2. Bottom: Ramp profile of dipole field (left), and development of rms of particle distribution (right) in the variables u (red) and v (black).